Towards a computational approach for Chabauty method

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Topics

- A Rational Introduction
- What is the Coleman-Chabauty Method?
- What is a Coleman Integral?

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- $\#X(\mathbb{Q})$ is related to g
- if $g = 0, 1$ then $\#X(\mathbb{Q})$ can be infinite
- what if $g > 1$?

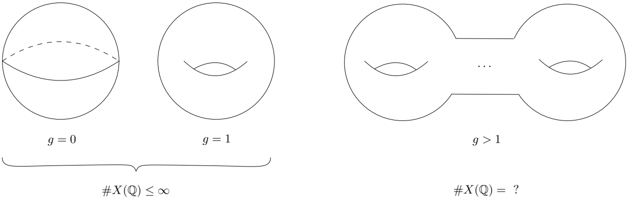

Figure:

Non-effective results

Falting's theorem, 1983

Let K be a number field and X a nice curve over K of genus g. If $g > 1$, then $\#X(K) < \infty$.

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How to compute effectively $X(\mathbb{Q})$?

Hirowaka-Matsumura's question

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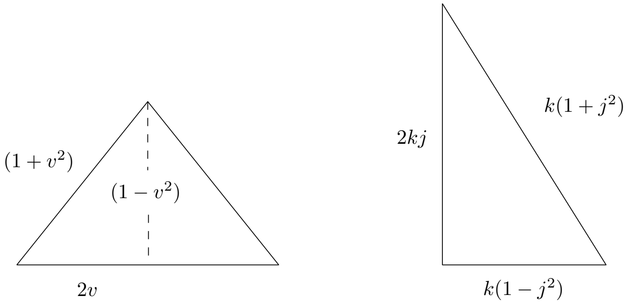
Does there exist a rational right triangle and a rational isosceles triangle that have the same area and the same perimeter?



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Let $x = v + 1 \implies \exists x \in \mathbb{Q} \cap (0, 1/2)$ s.t.

$$2xk^2 + (-3x^3 - 2x^2 + 6x - 4)k + x^5 = 0$$

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Goal: determine $X(\mathbb{Q})$!

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- X embeds into J
- $J(\mathbb{Q})$ is finitely generated abelian group $\implies$ $J(\mathbb{Q}) = J(\mathbb{Q})_{tors} \oplus \mathbb{Z}^r$

where $rk(J) := r$

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Magma Implementation of the 2-descend

$> R < x >:= PolynomialRing(RationalField());$

$> X := HyperellipticCurve(x^6 + 12 * x^5 - 32 * x^4 + 52 * x^2 - 48 * x + 16);$

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$> RankBounds(J);$

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output of RankBounds is a lower bound on rank, followed by an upper bound on rank $\implies r = 1$.

Chabauty's Thm, '41

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and the effective version

Coleman's Thm, '85

$X/\mathbb{Q}$ nice curve s.t. $g \geq 2$, $r < g$, $p > 2g$ for p a prime of good reduction $\implies \#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2$.

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After a search

$$X(\mathbb{Q}) = \{\infty^\pm, (0, \pm 4), (1, \pm 1), (2, \pm 8), (12/11, \pm 868/11^3)\}$$

$$\implies \#X(\mathbb{Q}) = 10!$$

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Hirakawa–Matsumura's Theorem

Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area.

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Hirakawa–Matsumura's Theorem

Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area.
The unique pair consists of the right triangle with sides $(377, 135, 352)$ and isosceles triangle with sides $(366, 366, 132)$.

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Theorem 2 (Katz,Rabinoff, Zureick-Brown)

If $X/\mathbb{Q}$ is a nice curve with $r \leq g - 3 \implies$ $\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$.

Not always so lucky!

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www.lmfdb.org/Genus2Curve/Q/971/a/971/1

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- J is simple and $rk(J) = 1 \implies J(\mathbb{Q}) \simeq \mathbb{Z}$
- C has good reduction at $p = 3$ and $\#C(\mathbb{F}_3) = 7$

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Katz and Zureick-Brown extended Stoll's result to the case of bad reduction.

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We suspect we have all of the $\mathbb{Q}$-points and we would like to prove this!

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- $\mathcal{I}|_{X(\mathbb{Q})} = 0$ but $\mathcal{I} \neq 0$, with fin. many zeros
- $\int$ p-adic integral
- regular 1 form ω s.t. $\forall P \in X(\mathbb{Q}),\ \int_\infty^P \omega = 0$

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$X(\mathbb{Q})$ is contained in a finite, computable set $\implies$ compute $X(\mathbb{Q})$ plus something hopefully small!

Coleman's Effective Chabauty

Reminder

- $\omega \in \Omega^1(k)$ is *regular* is $\forall P \in X(\bar{k}),\ \ \nu_P(\omega) > 0$
- ω is of 2nd kind if it has residue zero $\forall P \in X(\bar{k})$

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Reminder

X^{an} is the *rigid analytic* space over $\mathbb{Q}_p$ associated to $X/\mathbb{Q}_p$. There is a specialization map from $X^{an} \to X$ mod p. The fibers of this map are called residue disks.

Theorem 3 (Coleman Integral)

$X/\mathbb{Q}_p$ nice curve. The p-adic integral $\int_P^Q \omega \in \overline{\mathbb{Q}}_p$ defined $\forall P, Q \in X(\overline{\mathbb{Q}}_p), \forall \omega \in H^0(X, \Omega^1)$ is s.t.

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- *D principal $\implies \int_D \omega = 0$*

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  For $0 \neq \omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$

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- (not trivial!)

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additive in Q, linear in ω s.t.

$$\langle [D], \omega \rangle = \int_D \omega.$$

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We define

$$A := \{\omega \in H^0(X, \Omega^1) | \, \forall P \in J(\mathbb{Q}), \; \langle P, \omega \rangle = 0\}$$

as the subspace of *annihilating* differentials.

$i\colon X \hookrightarrow J$ induces $H^0(J_{\mathbb{Q}_p}, \Omega^1) \simeq H^0(X_{\mathbb{Q}_p}, \Omega^1)$

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which induces the homomorphism

$$\log\colon J(\mathbb{Q}_p) \to H^0(J_{\mathbb{Q}_p}, \Omega^1)^*$$

Thus we get

$$\begin{array}{ccc}
X(\mathbb{Q}) & & \\
\downarrow & & \\
X(\mathbb{Q}_p) & \xrightarrow{AJ_b} & \\
\downarrow & & \\
J(\mathbb{Q}_p) & \longrightarrow & H^0(J_{\mathbb{Q}_p}, \Omega^1)^* \simeq H^0(X_{\mathbb{Q}_p}, \Omega^1)^*
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By construction $X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_1$

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- compute

$$\alpha_i = \int_{(0,1/2)}^{(-1,-1/2)} \omega_i$$

Using SageMath we get

- $\alpha_0 = 3+3^2+3^4+3^5+2\cdot3^6+2\cdot3^7+2\cdot3^8+3^{10}+O(3^{11})$
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$$\eta = \alpha_1\omega_0 - \alpha_0\omega_1 \in A$$

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where t runs over all residue disks.

Then we solve $\forall z \in X(\mathbb{Q}_3)$ s.t.

$$\int_{(0,1/2)}^{z} \eta = 0.$$

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So we need to compute Coleman integrals between points not in the same residue disk!

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A *wide* open subspace of X^{an} is the complement of the union of a finite collection of disjoint closed disks of radius $\lambda_i < 1$.

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Consider

- $a, b \in \overline{\mathbb{Q}}_p$
- $P, Q, R \in U(\overline{\mathbb{Q}}_p)$
- $\xi, \eta \in \Omega^1(U)$ for U wide open subspace of X^{an}

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- *linearity:*

$$\int_P^Q (a\eta + b\xi) = a\int_P^Q \eta + b\int_P^Q \xi$$

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- *for ω defined over $\mathbb{Q}_p$, we have $\int_P^Q \omega \in \mathbb{Q}_p$*

More Coleman Integration

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- change of variables: for $\omega' \in \Omega^1(U')$,

$$\int_P^Q \varphi^* \omega' = \int_{\varphi(P)}^{\varphi(Q)} \omega'$$

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Goal

Integrate

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Using p-adic heights it is possible to integrate also forms of the 3rd kind.

Sketch of Explicit Coleman Integration

- φ lift of p-Frobenius from the special fiber

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- solve $\int_P^Q \omega_i$

Back for the last time to

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To resume:

- $C(\mathbb{Q})_{known} = \{\infty, (0, \pm\frac{1}{2}), (\mp 1, \pm\frac{1}{2})\}$
- $C(\mathbb{F}_3) = \{\infty, (0, \pm 1), (1, \pm 1), (2, \pm 1)\}$
- $$\int_{(0,1/2)}^{P_t} \eta = \int_{(0,1/2)}^{P_0} \eta + \int_{P_0}^{P_t} \eta$$

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- now the computation is purely local

We carry out the computation in the residue disks of

$$\infty, \left(0, \frac{1}{2}\right), \left(\pm 1, \frac{1}{2}\right)$$

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$$C(\mathbb{Q}_3)_1 = C(\mathbb{Q})_{known} = C(\mathbb{Q})$$

A Couple of Perspectives

- Quadratic Chabauty
- Computations and Algorithms for Q.C.

Essential Bibliography

Notes from Arizona Winter School 2020:

- Jennifer S. Balakrishnan, J. Steffen Müller, "Computational Tools for Quadratic Chabauty"
- David Zureick-Brown "Abelian Chabauty"

Thanks for the attention![1]

[1] I deeply thank Dr. Yelena Yuditsky for the precious help with the drawings.